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THE RELATION OF SPACE AND GEOMETRY TO EXPERIENCE*

I. Geometry as a Schematization of Experience

GEOMETRY is considered by every one to rank among the most certain of sciences. One can have grave doubts, for example, as to the universal validity of any theory in biology, or even honest misgivings concerning the absolute precision of the law of the conservation of energy, but it is hard to imagine a man who is really sincere in questioning the theorem of Pythagoras, that the square on the hypotenuse of a right-angled triangle is equal in area to the sum of the two squares on the legs of the triangle. This conviction which we possess that the theorems of geometry are valid seems essentially independent of any confirmation or substantiation by experience. After we are really initiated into the processes of geometrical reasoning, our certainty of the truth of the theorem of Pythagoras cannot be augmented nor diminshed one jot nor tittle by any actual measurement of a figure illustrating the theorem, if the figure should not substantiate the theorem, so much the worse for the figure, we should say.

It is a highly significant fact, however, that this very science of geometry, which seems to keep itself so independent of experience, is one of the most useful of all sciences in our daily life of experience. The surveyor, the navigator, the carpenter, all make continual use of geom-

* This sequence of lectures was read at Harvard University in the Fall Semester of 1915.

etry in the course of their every-day pursuits, and not only do they do so, but they have an implicit confidence, which always proves to be justified, that the results of their geometrical reasonings—provided only that these are correct in a purely intrinsic, geometrical sense and are based on correctly gathered data—will lead them to perfectly correct conclusions with regard to the world of things experienced with which they deal in their daily lives. The surveyor knows that if his observations are correct, and if he has committed no error of geometry in his computations, the map which he has designed in accordance with a few elementary geometrical laws will be a good map of the region it represents. We have thus the interesting spectacle of a science which seems to scorn experience as its basis, yet furnishes results of the utmost empirical application and value. The question at once occurs to us: How does this happen?

Several theories of the nature of geometry have been devised to bridge this gap. Let us first consider Kant's discussion of geometry. I do not propose to consider here the whole of Kant's treatment of this topic, but only a certain aspect of it—that aspect, namely, which is expressed in the following passage:[1] "Geometry is a science which determines the properties of space synthetically, and yet *a priori*. What, then, must be our representation of space, in order that such a cognition of it may be possible? It must be originally intuition. . . . But this intuition must be found in the mind *a priori*, that is, before any perception of objects, consequently must be pure, not empirical, intuition. For geometrical principles are always apodeictic, that is, united with the consciousness of their necessity, as, 'Space has three dimensions.' But propositions of this kind cannot be empirical judgments, nor conclusions from them."

[1] *Critique of Pure Reason*, Transcendental Aesthetic, §3, Meiklejohn's translation.

That is, Kant says not only that geometry is known *a priori,* but also that our whole original knowledge of *space,* the subject-matter of geometry, is *a priori,* and he regards these two assertions as practically tantamount to one another. It seems to the casual observer, however, as though spatial properties could also be given to us *a posteriori,* in experience. It seems as if the straightness of a stick or its length were known quite as empirically as its color or its hardness. Whatever we may say about space, there is no question possible with regard to the statement that spatial qualities are capable of being experienced. Now, it is not with space in any ulterior sense, but with *spatial qualities* that geometry, as used by the surveyor or the navigator, deals. It is not lines in any purely abstract meaning of the term, but the hair-lines in his telescope, or the path of a light-ray, that concern him, and he knows that if his measurements of the lengths, straightness, angles, etc., of these are correct, his computations will also be correct, provided only that he has made proper use of geometrical reasoning. It is such lines as these that form part of his space—and yet he feels the need of no experiment to substantiate the result of his geometrical reasoning. The *a priori* certainty which Kant attributes to geometry is one which is utterly irrelevant to its applications in our life; the abyss between his space, to which geometry applies, and the concrete spatial properties of concrete things, remains unbridged in his system, notwithstanding the fact that he calls space the form of our external experience since the apriority of the geometry which we apply must be the apriority of an empirical intuition, not that of a pure intuition. The geometry which he discusses is one which applies to an entirely non-empirical realm, and which he nowhere brings into touch with those fields of experience in which our every-day geometry plays so great a role.

One of the chief motives which leads Kant to this somewhat incomplete if not positively unsatisfactory treatment of geometry, as one can readily see from this paragraph which we have quoted, is that he considers the apriority of geometry impossible unless our knowledge of its subject-matter is also *a priori*. It is clear, then, that *if we can consistently hold that it is possible for geometry to be a priori, and yet to have an empirical subject-matter,* one strong argument in favor of Kant's view of space has vanished, and we are able to formulate a theory of the relation of the non-empirical science of geometry to the objects of our experience as surveyors or navigators, etc., which is more consonant with the views of our every-day common sense than that of Kant. It is this view of the relation between experience and geometry— the view, namely, that geometry, though *a priori,* deals with an empirical subject-matter— which I intend to suggest as a possibility in what follows.

Before I go on, however, to my discussion of this theory, I wish to devote a little time and attention to a third theory, different both from that of Kant and from that which forms the thesis of this course of lectures. This theory is that of Ernst Mach, as expounded in his little book, *Space and Geometry.*[2] Professor Mach's views form the precise antithesis of those of Kant, both with respect to space and to geometry. As to space, he says:[3]

"If for Kant space is not a 'concept,' but a 'pure (mere?) intuition a priori,' modern inquiries on the other hand are inclined to regard space as a concept, and in addition as a concept which has been derived from experience. We cannot intuit our system of space-sensations *per se;* but we may neglect sensations of objects as something subsidiary; and if we overlook what we have done, the notion may easily arise that we are actually concerned

[2] Translated by T. J. McCormack, Open Court Publishing Company, Chicago.
[3] *Op. cit.*, p. 34.

with a pure intuition. If our sensations of space are independent of the quality of the stimuli which go to produce them, then we may make predications concerning the former independently of external or physical experience. It is the imperishable merit of Kant to have called attention to this point. But this basis is unquestionably inadequate to the complete development of a geometry, inasmuch as concepts, and in addition thereto concepts derived from experience, are also requisite to this purpose."

Mach claims, in other words, that space is essentially a system of *space-sensations* or *space--experiences,* which seems to take the form of a "pure intuition" merely because in our geometrical considerations we confine our attention to one particular phase of the objects with which we are concerned, and neglect all those aspects of our experiences which, though they are necessarily present, are not spatial in their nature. According to him, he says, we are enabled thereby to consider the interrelations of the spatial aspects of our experience with entire disregard of what the other sides of our experience may be. Nevertheless, he holds, space is given to us in a completely empirical manner. Or, as Mach puts it in another book of his,[4] "Space and time are well-ordered systems of sets of sensations."

It seems obvious to the common-sense of us all that Mach is at bottom correct in this statement, for space is somehow or other, we all should say, a system of experiences. Everything looks promising, therefore, for a satisfactory account of the sources and nature of our geometrical certainty. Let us see what the explanation of this is which Mach offers us. He expounds his view as follows:[5]

"The knowledge that the angle-sum of the plane triangle is equal to a *determinate quantity* has thus been reached

[4] *The Science of Mechanics,* translated by T. J. McCormack, Open Court Publishing Co., p. 506.

[5] *Space and Geometry,* p. 58.

by experience, not otherwise than the law of the lever or Boyle and Mariotte's law of gases. It is true that neither the unaided eye nor measurements with the most delicate instruments can demonstrate *absolutely* that the sum of the angles of a plane triangle is *exactly* equal to two right angles. But the case is precisely the same with the law of the lever and with Boyle's law. All these theorems are therefore idealized and schematized experiences: for real measurements will always show slight deviations from them. But whereas the law of gases has been proved by further experimentation to be approximate only and to stand in need of modification when the facts are to be represented with great exactness, the law of the lever and the theorem regarding the angle-sum of a triangle have remained in as exact accord with the facts as the inevitable errors of experimenting would lead us to expect; and the same statement may be made of all the consequences that have been based on these two laws as preliminary assumptions."

This result—namely, that Mach regards the certainty of geometry as of empirical origin, and simply due to the fact that our experiments with lines and angles, etc., by means of paper-folding and similar methods have always substantiated our geometrical predictions as well as could be expected when we take into consideration the inherent inaccuracies of the experiments—this result, I say, is by no means satisfactory. Nobody would ever think of testing the theorems of Pythagoras by means of a foot rule or a protractor; the only things which would be tested by such an attempt and which would have to be rejected in case of a non-verification of the theorem would be the foot rule or the protractor. However useful paper-folding and similar pursuits may be in leading our interest toward things geometrical and in giving us the first dawning ideas about what it is with which geometry concerns itself, geom-

etry deals *directly* with points, lines, planes and angles, and not, except in some periphrastic sense, with such gross topics as folded bits of paper, rules, and micrometers. Whatever the edge of a piece of paper may do or be, a *line* is the shortest distance between two points, does not cut any other line in more than one point, and has all the other properties which are attributed to lines in a text-book of geometry. If a crease in a piece of paper fails to have these properties, why—it simply is not a line. However useful geometry may be in the theory of paper-folding or navigation or astronomy, *prima facie* geometry is *not* the study of paper-folding nor of navigation nor of astronomy, and the accuracy or correctness of any part of any of these studies may be impeached without involving as a corollary the impeachment of any portion of geometry or theorem belonging to it. The geometry of which Mach talks is simply not the geometry of the mathematician; Mach solves the problem of space and geometry to his own satisfaction by flatly ignoring the non-experimental nature of geometry, just as Kant solves it by not entering into a discussion of that empirical character which actually pertains to space. Both positions are unnatural; what is the natural alternative which avoids the objections besetting each of them?

I have already stated that the view which I maintain in this course of lectures is that geometry, though *a priori*, deals with an empirical subject-matter. How is this, however, possible? How can our study of a subject which is known in a manner open to all the uncertainties and inaccuracies which beset empirical knowledge in all its manifestations—namely, space—be possessed of an *a priori* and purely intrinsic certainty, not rooted at all in experience? The answer to this question is by no means as difficult as it might seem at first sight. It will be noted that Mach does not make geometry deal with raw, undi-

gested experience, but, as he says, "All these theorems are . . . idealized and schematized experiences."

Now, the study of an idealized or schematized experience differs from that of a raw or crude experience in that it has to take account of two distinct factors—the experience, and the mode of schematization employed. To illustrate how this is the case, suppose that I am considering a set of statistical tables of the death-rate of Boston from year to year. I may regard these tables from several different standpoints. I may be interested, for example, in the seasonal fluctuations of the death-rate. In this case the table of statistics gives me information which could not have been predicted with more than approximate accuracy and certainty, and which is completely dependent upon concrete experience. On the other hand, I may be primarily interested in the method of tabulating statistical data which is used in these tables; in this case, when I have once grasped the principle underlying the method, I am quite as well able to predict anything you please in the next year's tables which concerns details that are dependent solely on the method of tabulation employed as I am to yield the same information concerning this year's tables or concerning last year's tables. The method of tabulation employed may and should be made as suggestive as possible of the actual empirical laws of the death-rate of Boston and as useful as possible in the handling of the data tabulated, but once it has been chosen, it is entirely independent of the particular empirical properties of these data, and remains essentially incapable of substantiation or of contradiction by them. Thus, though the study of his tables from the standpoint of the form of tabulation employed is of immense practical use to the statistician for the handling of his empirical material, once that form is definitively fixed, it is really an *a priori* science, notwith-

standing the fact that the data expressed in the tables are themselves known *a posteriori*.

It is possible to regard geometry in a way quite parallel to a set of statistical tables—though I do not mean to suggest that statistics play any part whatsoever in geometrical reasoning. We may regard a point, for instance, not as a direct object of experience, but as a certain arrangement or collection of objects of experience, in a manner which I shall explain in detail in the subsequent lectures of the present course. A point of this sort will, in general, depend for its actual properties on the concrete natures of the experiences of which it is constructed, but it will also have certain properties which, unlike its other attributes, are independent of the concrete natures of these particular experiences, and are predictable on the basis of a knowledge merely of the principle in accordance with which the points of our space have been synthetized from our experience. These latter properties of points are studied in geometry, while those which are dependent on concrete experiences belong rather to physics or to the other natural sciences. Thus space, which is made up of points, lines, etc., constitutes a kind of tabulation of the experiences of our outer senses; yet geometry, which has space as its subject-matter, since it depends on the method of tabulation alone, as I claim in this course of lectures, is an *a priori*, not an experimental, science. This is the view for the possibility of which I am here pleading.

My view might be stated as follows: Geometry is the science of a *form* into which we cast our spatial experiences. I shall not express my view in this manner, for I wish to keep it clearly distinct from two other views which might with equal justice lay claim to this mode of expression. These views are that of Kant, upon which we have already touched, on the one hand, and the view of those mathematicians, on the other, who hold that the only spe-

cies of geometry which can possess *a priori* certainty is that geometry which concerns itself, not with the actual points and lines of the world in which we live, but with the *laws* in accordance with which a great many of the properties of these points and lines can be deduced from a small number of properties which they seem to possess, or at any rate seem to possess approximately.

Let us first see wherein our view differs from that of Kant. Kant says that geometry is the synthetical science *a priori* of the form of the external sense, whereas we say that geometry deals with the intrinsic properties of a schematism into which we cast our external experiences; wherein lies the real difference between these two very similar views, and what is its significance? The difference is this: Kant regards geometry as the study of a schematism imposed on the world by our external senses themselves, before any act of experience, and utterly independently of any such act. On the other hand, we maintain that geometry deals with an experience schematized after it has come into existence, and with concrete practical ends in view, even though this schematism may be permanent once it has come into existence and been accepted by us. As a consequence, Kant is unable, as we have previously indicated, to explain how it is that we are able to apply geometry to experience *in a certain concrete and definite manner,* as it is applied by the sailor and the surveyor, or at least he fails to give any hint of how this application is to take place, for the schematism which constitutes the subject-matter of geometry is made, he tells us, before and without reference to the concrete experiences of the surveyor and the sailor, by the essential nature of the outer senses, themselves, and would be the same were there no such particular experiences as those of the sailor or the surveyor. We, on the other hand, are able to maintain that the schematism of geometry is useful for the sur-

veyor and the sailor just because it is designed with the purposes of the surveyor and the sailor in view. This is still true even though that schematism remains just what it is forever, once it has been selected. For example, we choose the schematism "line" in such a manner that some particular line of geometry will be determined as unambiguously as possible in a certain easily recognizable manner by every ruler edge or plumb-line or line of vision in our actual experience. Then, while the lines we have chosen in our schematism may have a host of interesting and valuable properties which are determined by the schematism alone, we may make certain of our geometrical objects standing hostage, as it were, for the physical objects mentioned above, and make our reasonings and experiments refer to these lines rather than to the physical objects themselves, so that our reasonings and experiments may be facilitated by the manifold transformations and systematizations suggested by pure geometry. Since our geometrical lines, though constructions and schemata, are constructions and schemata made on the basis of concrete experiences, we are able to recognize empirically this correspondence between geometrical lines and certain physical entities to which we have just referred, and hence make the former take the place of the latter in the formulation of scientific laws. This cannot be done on the basis of Kant's theory—and this is its fatal defect—because space, according to him, though the form of our external experience, is completely prior to any concrete experience, and hence no correspondence between certain spatial entities and certain physical entities can be recognized empirically, if we accept his theory of the matter.

So much for Kant; let us now consider the pros and cons of the view of those who hold that the only sort of geometry which can possess *à priori* certainty is that geometry which concerns itself, not with the actual points and

lines of the world in which we live, but with the *laws* in accordance with which a great many of the properties of these points and lines can be deduced from a few laws which we observe that these points and lines possess, or very nearly possess. This view, that is, says that the real subject-matter of geometry is the formal deduction of its theorems from its axioms, which are not self-evident statements concerning the space in which we live, but mere hypotheses which may perchance be satisfied by an infinity of systems, and it claims further that geometry is not at all concerned with the question whether these axioms and theorems apply to any particular objects or constructions in the world of sense. This latter application, it maintains, must be determined by experience alone, and depends on experience for its validity. Now, it is perfectly true that there is a legitimate non-empirical science, which has as good a claim to the name of geometry as the discipline which we are discussing here, which is concerned with the deduction of the theorems of Euclid from the axioms of Euclidean geometry. I doubt, however, whether this mere abstract logical deduction constitutes the whole of what we ordinarily call geometry, or even the whole of that part of geometry which can lay claims to *a priori* certainty. There certainly appear to be such things as lines, which are more than mere blank spaces in the scheme of symbolism or of logical deduction by means of which the appropriate theorems are obtained from any set of truths which can be put into the form expressed in Euclid's axioms. It seems as if these lines must, from the very necessity of their nature, satisfy the laws of Euclidean geometry, while certain particular lines bear an intimate association with such concrete empirically known things as straight edges and light-rays. This association seems to be presupposed in our every-day life when we say, "This is more nearly a true line than that," as if the true line

were a sort of a criterion with which we could empirically compare certain empirical objects. This two-faced aspect of geometry, which is *a priori,* yet deals with an empirical subject-matter, is not explained by those who hold the view we criticize, and is explained on our view.

We hold, then, that geometry is an *a priori* science, which deals with a certain schematization of experience, which we may call space, in so far as its properties depend on the method of schematization alone. This schematization has a superficial appearance quite different from that of the experiences of which it is composed before they are schematized. Experience presents us only with objects that have extension, while a point has no extension. Experience never gives us a perfectly straight line, nor a precise circle, nor an absolutely accurate sphere. All these things, however, form topics dealt with in geometry. Now, we have claimed in this paper that geometry is a schematization of experience, not in the sense that it is a kind of approximate copy of experience with all the roughnesses left out, but in the sense that it is formed from experience by the application of some principle, just as a table of statistics represents the facts it concerns in accordance with a certain principle of tabulation. Just as, notwithstanding the fact that a table of statistics does not resemble the matters tabulated, a statement about the former is but a periphrasis for a statement about the latter, so a geometrical proposition is really concerned with experience, notwithstanding the fact that its direct subject-matter has an appearance differing in many respects from that of experience.

After all this talk of geometry as a method of tabulation, many of you will want to see a concrete example of this sort of tabulation, taken from the field of geometry. It is rather difficult, however, to exhibit such an example in the limited portion of this lecture which remains. I

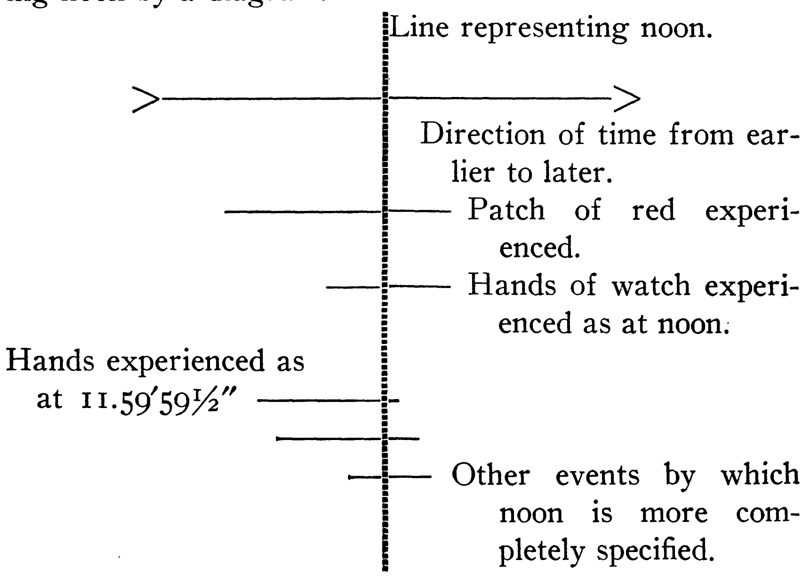
can, nevertheless, give you an example of a similar tabulation employed in a field very analogous to geometry—the study of the formal properties of time.

There are certain laws which we always unquestioningly accept as valid concerning time in quite the same spirit that we hold geometry to be *a priori*. We believe, for example, that time is composed of instants which are timeless, that no two instants are contemporaneous, that of two distinct instants, one must precede the other, and that if the instant *a* precedes the instant *b*, and the instant *b* precedes the instant *c*, that *a* precedes *c*. We consider these statements as quite as truly *a priori* as the theorem of Pythagoras, and regard the former and the latter as quite analogous with one another. The events which we experience, however, always occupy time, an event may neither precede nor follow another, and so on indefinitely. How are we able, the question is, to regard instants as tabulations of events of such a sort that we can be sure, from a knowledge of the method of tabulation alone, without any concrete empirical knowledge of events, that instants will have the formal properties we have attributed to them? I shall give such a method of tabulation in the following paragraphs, though I cannot spare the time to show, as is the case, that it is a consequence of the method itself that instants have the formal properties we have attributed to them.

Experienced events are said to happen at certain instants: what do we mean by such statements? When I say that I see this patch of red here at, say, noon, what do I mean? My first meaning is, perhaps, that I have taken out my watch, looked at it, and have seen both the hands

pointing at the figure XII, and that I have experienced this as simultaneous with my experience of the patch of red. But if I look into the matter more thoroughly, I find that this is not all I mean to assert when I say that I see this patch of red at noon. The time during which I see the hands of my watch in a certain position is always of a duration not zero. If the watch had suddenly passed out of existence while I was looking at it, I should still have continued to have seen it for a fraction of a second, during which noon would have passed beyond recall. Now, I can approach more nearly to a precise formulation of what I meant by the proposition that I saw the patch of red at noon if I name still other events which were experienced as simultaneous with the position of the hands of the watch, but which did not endure in experience for the whole period that the hands of the watch were experienced to remain in their position. For example, I can say, perhaps, that this patch was not only experienced as simultaneous with the position of the hands indicating noon, but that the experience of their indicating 11.59′59½″ had not yet died out while I saw the patch. By noting more and more events, each experienced as simultaneous with the patch, and each experienced as simultaneous with each other (for they all, we should say, are experienced as being at noon), we can finally arrive at the specification of a given instant without duration at which the patch was seen, though all the events used in fixing this instant may have consumed time, and have been possessed of all the other gross properties characteristic of *experienced* events, and the relation of simultaneity among them may have been given in experience.

Perhaps I can best illustrate our method of determining noon by a diagram.



We are able to continue this process further and further by the adjunction of more and more events to the set by which we determine noon. Every such event will be experienced as simultaneous with every other. Finally we shall come to a stage where no more new terms can be found which we can adjoin to our set—that is, there will be no other events which will be experienced as simultaneous with *all* the members of our set. In such a case, we shall have given as complete a determination of noon as is possible on the basis of experiences. The patch of red, which we wish to say is seen at noon, is one of these.

But what *is* noon, which seems to form the subject-matter of the proposition, "This patch of red is seen at noon," which we have been considering? We wish to interpret noon as a sort of tabulation of experience: the answer we give shall therefore read, "Noon is the whole class of events, each of which is experienced as simultaneous with each other, which contains every event experienced as

simultaneous with all its constituent events, by means of which we have dated our patch of red." This definition may seem to be circular, for it may seem that the class of events in question could not be specified except with reference to a pre-existing notion of noon. This criticism is, however, invalid, for it may be shown that if instants and events have the formal properties and interrelations that we universally attribute to them, every instant, such as noon, will determine uniquely and be uniquely determined by some set of events of which every two members are experienced as simultaneous, and which is such that it contains every event which is experienced as simultaneous with all its members. The definition of these latter entities involves no circularity, for it depends merely upon a previous acquaintance with events, the relation of simultaneity and a few elementary logical notions such as that of a collection, and not at all upon any specific acquaintance with noon, or with any other instant.

Now, the relation of simultaneity among events, whether as such it can be experienced or not, is certainly far closer to experience than an instant. In all this work, the point we are making is not that the terms and relations with which we start and which we take to represent experience are immediately given—we do not even assume that there are immediately given terms and relations—but that, if I may use such a phrase, they are closer to givenness, that they are less elaborated, that they are the results of a lesser degree of sophistication than the ordinary notions of science. Whether the "experience" with which we started in this lecture is itself already schematized or not does not concern us here; it is enough that space and time mark a degree of schematism greater in intricacy than what we here call experience.

To sum up: we have contrasted the aloofness of geometry from empirical verification with its tremendous value

when applied to experience, and have noted the problem which this situation creates. We have discussed Kant's views on the relation between geometry and experience, and have seen that his statement that geometry deals with space, which is given prior to any concrete experience whatever, is hard to reconcile with our empirical recognition of geometrical forms. We have seen that a view which should hold that geometry, though *a priori,* deals with an empirically known subject-matter, would avoid this particular difficulty. Then, taking up Mach's standpoint, it became clear to us that his view, that geometrical certainty is of experimental and empirical origin, is in direct conflict with the practice of all mankind in matters geometrical, and that we all should hold that any geometrical experiment was a test rather of the instruments of measurement used than of the geometrical theorem involved. We noticed the suggestiveness of Mach's notion of geometry as the science of a schematized experience, but saw that in the study of statistical tables, for instance, certain aspects of the study of a schematized experience may be independent of the matter schematized, and depend only on the form of the schematism. We held geometry to be of a similar nature. We observed that our view lent itself to the formulation, "Geometry is the science of a *form* into which we cast our spatial experiences," but we observed that such a formulation would also cover Kant's view that geometry is the study of the form of the external sense, and the view that geometry is merely concerned with the deduction of geometrical theorems from geometrical axioms, so that the certainty which we usually attribute to geometry is entirely dependent on the fact that it is a science of abstract deduction. Kant's view, we found, differs from our view in the fact that it makes the form of the external sense prior to all experience, and consequently cannot explain the empirical identification of spatial entities, while we hold that the

schematism with which geometry deals is imposed only after and on the basis of concrete experience. The view that geometry is only concerned with a certain deductive chain did not explain why we act as if geometry were the *a priori* study of a certain concrete system which we can apply to experience as a criterion of straightness or of circularity or of any similar geometrical property. We saw that geometrical propositions, though they seem to deal with such entities as points, lines, etc., are mere paraphrases for propositions about experience in some more direct sense. We finally gave an example of the sort of schematization or tabulation of which geometry makes use, taking this example from *Our Knowledge of the External World as a Field for Scientific Method in Philosophy,* by Mr. Bertrand Russell.

The remaining lectures will be devoted to a more or less tentative discussion of the details of the methods of tabulation and schematization used in geometry. They will very often involve the use of simple geometrical reasoning, but only to prove that the methods of tabulation we here employ yield results similar to those yielded by the methods of schematization which we must tactily use in building up the entities of our every-day geometry.

II. The Point as a Tabulation of Solids

In our last lecture we put forward the view that geometry is concerned with the study of a certain tabulation or arrangement of experience, in so far as this arrangement is determined, not by the nature of the material arranged, but by some already fixed principle of arrangement. As an example of what such a tabulation or arrangement would be like, we gave a brief discussion of the definition of instants as arrangements of simultaneous events. In this and the following lectures we shall attempt, in a similar manner, to define the subject-matter of every-day geometry as a system of tabulations of things which can be experienced and their relations—that is, to exhibit the methods of schematism employed in geometry. In our last lecture we made the further claim that the ordinary theorems of Euclidean geometry could be regarded as consequences solely of the methods of tabulation and arrangement employed in geometry. In the ensuing portion of this course we shall try to show, as far as we are able, how points and the relations between points may be so defined as complexes of objects which can be experienced and of their empirically knowable relations that, though space will be dependent on sense, the geometrical properties of space will be independent of sense, and will follow solely from the schematism by which space is obtained from sense. We shall aim to show that, just as a cube does not depend for its cubical properties on the material from which it is made, just as a wooden, a stone and an iron cube all have eight apices, twelve edges and six faces, so a geometry, although its propositions may have relevance to the actual world in which we live, has a validity independent of the particular

nature of the world to which it applies. We maintain that geometry has this universal validity, not only in the sense in which it says that *if* any system satisfies a certain set of premises geometry is applicable to it, but also in the sense in which it asserts categorically that geometrical theorems must apply to the entities which we define as points, whatever the concrete nature of the world in terms of which they are defined may be.

The first task which we have before us is the determination of the fundamental spatial experiences in terms of which our subsequent schematizations and definitions are to be made. The first essential condition which these fundamental experiences must satisfy is that they should be genuine experiences. This excludes at once the possibility that they should deal with such essentially non-experienceable entities as points without magnitude or curves without thickness and so forth. It demands that the fundamental spatial experiences should concern such things as visible patches of color or tangible solids. This necessary condition which these experiences must satisfy leaves us still a great possibility of ambiguity as to their nature. As we are humanly unable to do what is perhaps the most natural thing in this situation and make our method of schematization apply to all experiences which we should ordinarily claim to have a spatial import, on account of the immense technical difficulties such a task would involve, we are obliged to introduce a certain degree of arbitrariness and artificiality into the selection of the fundamental experiences from which we shall build up our geometry. Whether the experience of the solid be primitive in experience or not, this much is certain, that it belongs to a much lower stratum of schematization and synthesis of experience than such unextended things as points, lines and other geometrical entities, and that things or solids are the last word in primitiveness and immediate givenness

for the man unsophisticated by psychology. Since our discussion in these lectures is only tentative anyway, and since solids offer a very convenient starting-point for the development of a schematism leading to geometry, we shall regard our primitive spatial experience as one dealing with solids. We have not yet completely specified the nature of our primitive spatial experience, however, as it is possible that there are many different kinds of facts concerning solids which can be experienced. One of the simplest to handle—although possibly not one of the simplest in the order of experience—of these facts is the fact that a certain solid is observed to contain a part in common with another solid. We shall, therefore, select an experience of the intersection or overlapping of two solids as the fundamental spatial experience. Two solids, we shall say, are experienced as intersecting or overlapping or having a part in common with one another if they both seem to contain some solid or if one seems to contain the other or if they seem to come into contact.

The experience of the overlapping of solids is not, however, as it stands, a sufficient point of departure for a schematization which is to lead to geometry. We wish to be able to define a straight line as a sort of a tabulation of solids. Now, if all we know about solids is the relations of overlapping that hold among them, we will be unable to discriminate between a straight line and a tortuous one. The whole of space could be kneaded like a lump of putty without changing a solid into anything else or altering the relations of overlapping which solids bear to one another, but by such a transformation you could deform a straight line into a curve as tortuous as you please. It is obvious, then, that if we are to be able to define straight lines in terms of the experience of the intersection of solids, we must put some kind of a limitation on the kind of solids considered. We shall put upon them the limitation that

they are to be *convex*. Now, a convex solid is one such that any two points which belong to it can be connected by a piece of a line which nowhere passes outside of it. Thus a solid sphere is convex, a cylinder is convex, a cone is convex, and a cube is convex, while a solid in the shape of an hour-glass is not convex, a doughnut is not convex, a bowl is not convex, and no figure which is hollow is convex. As a matter of fact, convexity is synonymous with the absence of hollowness in any sense, and since hollowness can roughly be judged by the eye and the finger without reference to straight lines, convexity may also be determined by a more or less direct reference to experience. We know what it means to say that a bowl is hollow and that a billiard-ball is convex long before we ever think of correlating these properties of solids with the definition of convexity just given. We can, therefore, make our fundamental experience that of the intersection of *convex* solids, and be sure that it is near to genuine experience. Further, it may be shown by a simple bit of geometry that if the world were, say, made of clay, and were so squeezed out of shape that all convex solids and their relations of overlapping should remain unchanged, every straight line would remain straight. Consequently, once the set of all convex solids in space has been identified, the set of all lines in space is determined, and it seems very probable, to say the least, that lines can be defined in terms of convex solids.

The sort of fact from the schematization of which we shall obtain space is, "This convex solid is *experienced* to intersect that one," and not simply, "This convex solid intersects that one." The formal properties of experienced intersection and of actual intersection are probably, however, closely analogous in most respects. Each solid may be regarded as having, outside of its physical extension, a sort of *aura*, of definite extent, such that two solids are experienced as overlapping when, and only when, the solids

formed out of each by adjoining to it its aura actually intersect. For example, two spheres a hundredth of an inch apart may seem to be in contact, as far as our unaided senses can tell: then we shall say that the aura of each extends at least one two-hundredth of an inch beyond its physical extension. The difference between the relation of apparent or experienced intersection among convex solids and that of their actual physical intersection is to all intents and purposes, then, a difference in the solids chosen as intersecting rather than in the formal properties of the relation of intersection itself, for if we replace convex solids by convex solids plus their auræ, we can interpret the apparent intersection of the former as the actual intersection of the latter.

We are now in a position to define our points—that is, to exhibit them as tabulations of convex solids. We shall define our points as collections or aggregates of solids. This may seem curious to many of you. "What!" you may think, "Is not a point small and a solid large? Is not a class of solids even larger than a solid? Then how can a point be a class of solids? How can the part be greater than the whole? How can points be made of solids, as you say, and solids also be made of points, as the mathematician says?" Now, all these questions result from a confusion of the relation of a member of a collection to the collection of which it is a member with the relation of an object filling a given space to an object filling a space including that which the first object fills. One tends to think, for example, that because Harvard University is a class of men, Harvard University fills more space than a single man. But, when one comes to think of this example more thoroughly, one sees that *in the sense in which a man fills a certain space,* it is nonsense to talk about Harvard University as filling any space. It is only in a metonymous sense that Harvard University can be said to fill the space occu-

pied by all its members. Harvard University has only such properties as belong to different logical dimensions from those of its members. In fact, it is a general proposition of logic that no collection can have any properties that can in precisely the same sense be significantly asserted—or denied, for that matter—of any of its members. Thus, Harvard University, although it has certain intimate connections with certain portions of space, cannot be said to occupy any space at all in the sense in which I now occupy, the space vertically above this platform, and in an analogous way, in the sense in which a solid can occupy space, a class of solids cannot occupy space, and in the sense in which a class of solids can occupy space, a solid cannot occupy space. It is, therefore, nonsense to speak of a class of solids as either smaller or larger than a solid. Hence we do not, in defining a point as a class of solids, make the part larger than the whole, for the point and the solid are rendered by such a definition incomparable as to magnitude.

The second paradoxical feature of our definition of a point—that we define a point as a collection of solids, whereas in ordinary geometry, a solid is regarded as a class of points—is eliminated still more easily. A solid, in the sense in which points are classes of solids, is an entirely different thing from a solid, in the sense in which a solid is a class of points. They are no more identical than the collection of clubs to which John Smith belongs is identical with John Smith himself. The only thing that entitles us to call both solids is that the world in which we live is probably so organized that corresponding to each solid in our first sense there is a class of points uniquely determined by it and representing no other solid than it, which we may call "the same solid as it," just as it might be that in some town one could identify every man by the list of clubs to which he belongs, and could say, whenever one should

see a list of clubs to which some man belongs, "That's John Smith," or "That's William Jones," or whoever else it might be.

Our definition of points in terms of solids is to be justified, as are all definitions in this kind of work, by its fruits. We shall so define a point that if the things we call convex solids are really the convex solids of an ordinary Euclidean space, the things we call points will correspond in a certain determinate manner to the points of ordinary Euclidean space; the things we shall later call lines will have all the nice properties that lines should have; and finally, the whole space we shall obtain as the end of our discussion will have all the attributes that pertain to our every-day space. On the basis of this first definition of points and of lines we shall give a second and finally a third definition of points and of lines which will, on the one hand, make each point of the first sort determine a single point of the second or third sort and each line of the first sort determine a line of the second or third sort in such a manner that the geometrical properties of a figure made up of points and lines of the first sort will be substantially unchanged if each point and line of the figure be replaced by the analogous point or line of the second or third sort—which will, I repeat, do all this *if the points and lines of our original system form a set satisfying the axioms of ordinary geometry*, or, to put it in a more elementary manner, if two lines in our first sense have a point in common when and only when two decent and well-behaved lines ought to have a point in common. On the other hand, we shall so frame our definitions of points and of lines of our third kind that, however irregular the formal properties of the points and lines of our first sort may be, however often lines that should intersect, did our original system obey the laws of geometry, fail to intersect, or lines that should fail to intersect do

intersect, our lines and points of the third sort must, so long as logic is logic, have all the properties appertaining to lines and to points in a Euclidean geometry. Furthermore, we shall develop a theory of measurement in this third space that we finally attain which will be consonant, on the one hand, with our usual ideas of the operations performed in actual physical measurement, and which, on the other, will be in perfect harmony with the laws of measurement laid down in ordinary Euclidean geometry. All this is done on the basis of our original definition of a point, and constitutes an ample justification for it.

After this rather long-winded apology for the definition of a point as a class of solids, let us state this definition in precise terms. *A point is a collection of convex solids such that (1) any two convex solids belonging to it are experienced as intersecting, and (2) if a convex solid is experienced as intersecting EVERY member of such a collection, it can only be itself a member of the collection.* We saw previously that the relation of experienced intersection among convex solids reduces itself to the relation of actual intersection among other solids—namely, those formed out of convex solids by adjoining their auræ to them, or as we shall hereafter call them, *a-solids.* Our definition is therefore practically equivalent to one which should read as follows: a point is a class of a-solids such that (1) any two members of the set intersect, and (2) any a-solid that intersects every member of the class must itself be a member of the class. Now, what does this mean?

Let us consider the class of all the a-solids which, as we should say in our every-day life, contain a given point x on their surface or in their interior. In the first place, every two members of this set intersect, for earlier in this lecture we have taken the term intersection to cover contact or tangency, and two figures with a point in common

either intersect bodily if the point in question lies in the interior of one of them, or come into contact with one another if the point lies on the surface of each of them. In the second place, it may readily be shown that if an a-solid intersects every a-solid that contains x, it must itself contain x. This proof depends upon the fact that if an a-solid does not contain a given point, another a-solid can be found which contains the point, but does not intersect the first a-solid. Taking this principle for granted—its truth can very easily be established on the hypothesis that the aural layer of a convex solid is of a uniform thickness throughout space, or on many similar hypotheses which do not assume so much—the desired consequence follows in this way: if an a-solid intersects every a-solid that contains x, but does not itself contain x, we get a contradiction, for by the principle which we have just enunciated, there must be a second a-solid, not intersecting our first a-solid, but containing x, while, by hypothesis, this is impossible. Consequently, if an a-solid intersects every a-solid that contains x, it must itself contain x. We have thus shown that a collection of all the a-solids which, as we should ordinarily put it, contain some point, satisfies both the criteria which a class of a-solids must fulfil to be a point by our definition, since any two members of it intersect, and any a-solid which intersects all its members belongs to it.

To give a completely satisfactory justification of my definition of a point as a class of a-solids whereof any two intersect and which are such that any a-solid intersecting every member of the set belongs to the set, it is not enough to show, as I have just shown, that every collection of all the a-solids containing some point, which may be said to represent or even to be that point, is a point in accordance with our definition; we must also show that no other collections of a-solids are points in accordance with our defini-

tion. We must show that if, on the one hand, a collection of a-solids does not exhaust those which, as we should ordinarily state it, contain some point in common, or if, on the other, there is no point common to all its members, the collection of a-solids in question fails to satisfy one or both of the two criteria which determine whether a given collection of a-solids is or is not a point in accordance with our definition of a point. Now, it is easy enough to show that if all the members of a collection *a* of a-solids contain a given point, but do not exhaust the collection of the a-solids which contain the point, there are other a-solids—i. e., the other a-solids containing the point in question—which do not belong to the collection *a,* but intersect every member of *a,* so that *a* is not a point in accordance with our definition. It is not easy to show, however, that if a collection of a-solids is of such a nature that there is no point, to use ordinary geometrical language, which all its members contain in common, this collection of solids fails to satisfy at least one of the two criteria both of which a collection of a-solids must satisfy if we are to call it a point in accordance with the definition we have given. In fact, I have not yet succeeded in proving this theorem, and I have nowhere seen any proof given for it, yet I am convinced that it is true and that it can be proved. I am convinced of this because, notwithstanding a considerable amount of effort, I have been unable to discover a single collection of a-solids, except the collection of all the a-solids that contain some given point, which satisfies both of the two conditions which all the things that are points by our definition must satisfy. Therefore, notwithstanding the gap in my chain of reasoning, I shall go on from this point as if I had proved that our definition of a point is perfectly adequate, and that the only collections of a-solids which satisfy our definition of a point are such as are made up from all the a-solids which,

as we should naturally put it, contain some point. If we suppose that this is proved, *provided that our experience of the relation of intersection among convex solids is to receive the geometrical interpretation in terms of a-solids which we have given it,* since our first definition of points in terms of the experience of the intersection of convex solids will then be practically equivalent to our second definition of a point in terms of a-solids, our points in our first sense, though defined in terms of an experience, will well deserve the name of points.

Our next task is to define what is perhaps the next most fundamental notion in geometry—the notion of a line—in terms of our experience of the intersection of convex solids. It will be remembered that convex solids stand in a very close relation to straight lines, for a convex solid is one that contains the whole of a bit of any straight line whose ends lie inside it. Now, this fact enables us to define a bit of a straight line in terms of our experience of the intersection of convex solids as follows. We have just seen how a point may be regarded as a class of a-solids which is what we should ordinarily call the class of all the a-solids containing that point. An assumption which we shall make at this point is that all a-solids are convex and that we can thus regard a point as a class of all of a certain kind of convex solids which contain a given point. This assumption is extremely natural. It is a consequence of the other assumption which we suggested previously, to the effect that the aural layer of a convex solid is of uniform thickness throughout space, but does not presuppose the latter assumption. From the hypothesis we have stated we can readily draw the conclusion that if a and b are any two points *qua* classes of a-solids, then every a-solid which forms a member both of a and of b contains, in ordinary geometrical phraseology, the whole piece of a straight line intercepted between a and b. That this is true follows from

the fact that *a* and *b* are points inside any a-solid which belongs to them both, since a member of a point is an a-solid which contains it. Consequently, since an a-solid is convex, any a-solid which belongs both to *a* and to *b* contains the whole linear segment or bit of line between them, and consequently every point on this segment. Therefore, every point on this segment possesses as a member any a-solid within which *a* and *b* lie. This is another application of the principle that the members of a point are the a-solids which spatially contain it. It is thus a necessary condition if *c* is to lie on the linear segment between *a* and *b* that all those a-solids which belong both to *a* and to *b* should also belong to *c*. That this condition is also sufficient may be proved on the hypotheses that the thickness of the aural layer of all a-solids is constant and that an a-solid can be transported to any part of space, and yet remain an a-solid. Both these hypotheses are very probably true—at least within that part of space whereof we have any experience at all. The deduction of the sufficiency of our condition from these hypotheses, though easy, is a little too intricate for us to give here.

We have, then, given a necessary and sufficient condition that one point, *qua* class of a-solids, should lie on the bit of line between two other points of the sort. Let us reinterpret this statement in terms of points consisting, not of a-solids, but of general convex solids. If three points, *a, b* and *c,* consisting of a-solids, are so arranged that *c* lies on the linear segment between *a* and *b,* and if *a′, b′*, and *c′* are, respectively, the points consisting of general convex solids corresponding to *a, b* and *c*, then it will be natural for us to say that *c′* is between *a′* and *b′* and on the line determined by them. That is, *c′* will be between *a′* and *b′* when and only when *c* contains all the a-solids common to *a* and to *b*. Now, *a* contains a given a-solid as a member when and only when *a′* contains the convex solid

from which this a-solid is formed by the adjunction of its aura, and a similar relation subsists between b and b', and between c and c'. Therefore, c contains all the a-solids common to a and b when and only when c' contains all the convex solids common to a' and to b'. Consequently c' lies on the linear segment between a' and b' when and only when c' contains all the members common to a' and to b'. Now, we have not yet defined linear segments or any such things, and this property of a', b' and c', when c' contains the common part of a' and b', is defined in purely logical terms introducing only such notions as those of part and class, involving no concrete geometrical notion, except such, of course, as are involved already in the notion of a point, which we have already defined in terms of our experience of the intersection of convex solids. We may therefore *define* a point c' to lie between two others, a' and b', when and only when c' contains the common portion of a' and b', and we shall be sure, on the one hand, that if our experience of the intersection of convex solids has the properties that are to be expected of it, this relation of betweenness will not have been misnamed, and, on the other, that this definition involves no notions other than that of our experience of the intersection of convex solids and certain general logical notions.

I wish now to define the notions of segment, end-point and line, in terms of the relation of betweenness just defined, and hence ultimately in terms of our experience of the intersection of convex solids. If a and b are distinct points, the class of all the points c which are such that c is between a and b constitutes the linear segment ab, and a and b are its end-points. The *line* ab is the class of all points belonging to linear segments which have at least two points in common with the linear segment ab. The agreement of all of these notions with the conventional

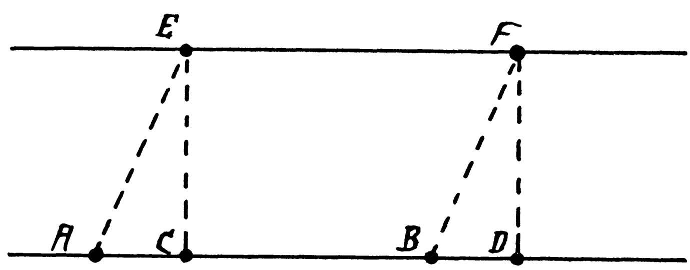
notions of segments, end-points and lines, subject to a certain reservation which we shall make in the next two lectures, will be obvious on a brief reflection. The adequacy of our definition of a line will be apparent if we reflect that any two linear segments which have two points in common are segments on the same line, while if x is any point on a line l, and s is any segment on l, a segment t can be discovered which will contain x and have at least two points in common with s.

To sum up what we have said in this lecture, we first defined a point as a class of convex solids, whereof any two are experienced to intersect, and which is further such that it contains as members all those convex solids which are experienced as intersecting all its members. We justified this definition of a point and showed that the entities which are thus defined as points are such things as one could naturally call points, providing that our experience of the intersection of convex solids has such formal properties as one would naturally attribute to it, since under this hypothesis each of our points will be a collection of all the convex solids which are experienced as containing some point, and may, since the notion of a point is only now defined for the first time, be identified with the latter point, which they are experienced as containing. We have defined a point a as between a point b and a point c when a contains the common part of b and c. From this definition alone we have derived definitions of a linear segment, of the end-points of a linear segment, and of a line. All these definitions have been made solely in terms of the experience which we have chosen as fundamental—that of the intersection of convex solids.

The work in this lecture is based on that of Dr. A. N. Whitehead and Mr. Bertrand Russell on space and time, as given in Mr. Russell's *Scientific Method in Philosophy,* Chapter IV. The definitions of betweenness and of a line are borrowed from Prof. Huntington's article in the *Mathematische Annalen* for 1912, but go back to the work of Kempe and Prof. Royce.

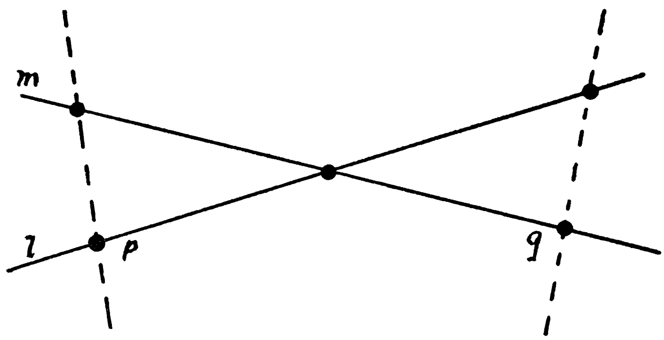
III. The Extension of Space Beyond the Bounds of Experience

In our last lecture, you will remember, we arrived at the definition of a point as a class of convex solids, and of a line as a class of points, in terms of our experience of the intersection of convex solids. These definitions, however, and, indeed, any definitions that start directly from our experience of the intersection of convex solids, must suffer from certain rather obvious defects. We intend to use our definitions of lines and of points to set up a theory of spatial measurement. To do this, we shall make much use of the construction of parallelograms: for example, we shall define the distance AB on a given line as equal to the distance CD on the same line if it is possible to construct a linear segment or piece of a straight line EF parallel to AB in such a manner that AE is parallel to BF and EC is parallel to FD. The following diagram will represent such a situation.



This demands that we are in possession of a definition of parallelism. In Euclidean geometry, to say that two lines are parallel is equivalent to saying that they lie in the same plane and do not intersect one another. We may

define two lines as being in the same plane if and only if they both have a point in common with each of a pair of intersecting lines l and m, and do not pass through the point of intersection of l and m; thus in the following diagram, p and q are in the same plane, or, as mathematicians say, are coplanar.



We might, therefore, define two lines as parallel if they are coplanar and do not intersect, without introducing any new fundamental notions into our system. But, as we have said, there are certain defects inherent in the definitions of lines and of planes that we have already given and these defects make such a definition of parallelism undesirable. These defects are due to the fact that our experience of the intersection of convex solids does not record the intersection of convex solids at the uttermost confines of space; beyond a certain extreme distance, whatever it may be, the intersection of convex solids is not experienced. As a matter of fact, I can hardly be said to have any experience of the intersection of convex solids except in the immediate neighborhood of my own body. As a consequence, any lines which would naturally be said to meet at a point lying outside the very limited region within which convex solids are experienced to intersect would, in accordance with the definition which we are considering, be parallel lines, for, by our definition of a point, a point is a collec-

tion of convex solids of which every two are experienced as intersecting, and which contains every convex solid experienced as intersecting all its members, and consequently there can be no points outside the region of the apparent intersection of convex solids, so that any two coplanar lines which fail to have a point in common within this region must fail to have any point whatsoever in common, and must, by the definition which we are considering, be parallel. Indeed, since our points are only such points as lie within a certain region of space, the lines which we defined in our last lecture are merely such parts of lines as lie within this region. Not only would this whole condition of affairs not be consonant with our natural notions of lines and of parallelism, but it would further fail to give parallelism even the most important formal properties which it possesses in ordinary geometry. For instance, it would be impossible to use the parallelogram construction as a criterion of the equality of two different linear segments on the same straight line, as a very simple geometrical construction, for it may easily be shown that our definition will make all distances along a given line equal. Hence, our definition of parallelism is at fault, and we must look around for a new one.

In what direction are we to look for this new definition of parallelism? The defect of the definition that we have just rejected—that two coplanar lines without a common point are parallel—is due to the fact that two lines which ought to have a point in common do not always have a point in common, if we define points and lines in the manner indicated in the last chapter. If we are able, then, to introduce new definitions of a point and of a line such that every two lines which ought, as ordinary, common-sense, decently-behaved lines, to have a point in common, will have a point in common, we shall have brought the problem of finding a definition of the paral-

lelism of two lines a great deal nearer solution. This new definition of a point and this new definition of a line should, if possible, be made in terms of our previous definitions of a point and a line alone, without introducing any new notions. The question we now ask is, can new definitions of a point and of a line, which make two lines contain points in common just when they ought to, be made in terms of our previous definitions of a point and of a line?

Now, we have seen that our previous definition of points gives us only the points within a certain region of space. Let us assume, for simplicity's sake—for, though this assumption is almost certainly false, it is not, as such, essential to our further argument, and it enables one to picture in his mind's eye what I have to say much better than any other hypothesis—that our points, in the sense in which points have been already defined, are *all* the points in the interior of some closed convex solid, and that our straight lines are consequently all the segments of straight lines intercepted by the surface of this solid. How are we able to recognize in terms of the points and lines inside this solid the entities, whatever they may be, that we should naturally call points outside of the solid? The problem is closely analogous to that of the recognition that a certain set of astronomical observations from various points on the surface of the earth all are observations of a single star, even though that star is utterly inaccessible to us. The problem to which I allude is not that of recognizing the star as the same star from observations at different *times;* it is the far simpler one of discovering that many observations made at the same time pertain to a single object. Let us suppose, for example, that we have four observers, a, b, c, and d, all looking at a star or planet x from different points on the surface of the earth. How

do the observers *a, b, c* and *d* know that it is a single star or planet at which they are looking?

The whole and complete answer to this question would involve considerations which are irrelevant here; it is obvious that one of the things that our four astronomers must know, however, is that they are all looking at the same *place*—that, in other words the axes of their telescopes converge on one point, or if the object at which they are all looking is sufficiently far away to be considered, for all practical optical purposes, as at an infinite distance, they must know that the axes of their telescopes point, to all intents, in one direction, or to put it otherwise, that they are all parallel. This knowledge, moreover, must be attained and is attained independently of any direct knowledge the astronomers have concerning the point to which all the axes of their telescopes converge, for this point is exceedingly remote from them, and is known by them in no other way than by these very observations concerning which we are now trying to find out why it is that the astronomers regard them as observations of a single point. When the astronomer says that at such-and-such an instant this point of space has this or that property—as for example that of being occupied by a planet—all that he means or has a right to assert must concern the observations in which the telescope is directed towards this point, for if the observations should remain the same, but the whole remainder of the universe should be changed in any manner whatsoever, the astronomer would still be entitled to make the same assertion concerning this point as formerly. The knowledge of the convergence to *x* of the optical axes of the telescopes at *a, b, c,* and *d* is attained by a measurement of the angles which the lines between *a, b, c, d,* and *x* make with one another and a measurement of the distances of *a, b, c,* and *d* from one another. These observations do not require any direct knowledge of *x,* but only of the posi-

tions of the telescopes at *a, b, c,* and *d,* for if we know the latitude and longitude of *a, b, c,* and *d,* if we know the compass-bearing of each telescope—that is, whether it is pointing east or west or southeast or north-northeast-by-north, etc.,—and if we know the slope of each of the telescopes, we know all the angles which any two lines connecting two of the points *a, b, c, d,* and *x* make with one another, and the mutual distances of the points *a, b, c,* and *d.* What we realy talk about, then, when we discuss the position of the planet is the aggregate of the positions of the telescopes by which it is observed.

Another thing to notice is that if we know that the telescopes at *a, b,* and *c* are all directed to one point, and that the telescopes at *b, c,* and *d,* are all directed to one point, we know that the telescopes at *a, b, c,* and *d* are all directed to one and the same point. This is rendered obvious by a simple diagram. The significance of this fact will appear if we consider that if the telescopes at *a* and at *b* are directed at one point, and the telescopes at *b* and at *c* are both directed at one point, all three telescopes need not be directed at any single point. This also is shown by a diagram.

We are now able to return to the discussion of our real problem—the problem, namely, how we are to recognize points outside that convex region of space within which all the points that we have already defined are located in terms of the points and portions of lines lying inside this region. The portions of lines lying inside our convex region—i. e., the class of all the lines that we defined in our last lecture—are the exact analogues of the telescopes or the astronomers whom we have just discussed. Just as the astronomer's statements about the position of a star really concern the positions of certain telescopes, so propositions which seem to deal with points lying beyond the bounds of our experience really concern certain collec-

tions of our lines: namely, with such as are made up of all those lines that "point at" some point lying outside our region. As far as we are concerned, such collections of lines, since they correspond uniquely to the points at which they are directed or from which they spread out, may be regarded as *constituting* these points. This situation can easily be rendered obvious by a diagram.

This definition gives rise to many problems. In the first place, how is it possible to get along in a system in which some points—those within the region of space directly accessible to experience—are the elements of which lines are classes, while other points in space ZZ those not in that region directly accessible to experience—are classes of these self-same lines? In the second place, is it possible to define the property which a class of lines has when every member of the class is directed towards some given point beyond the bounds of experience in terms of that experience which we have taken as primitive—in terms, namely, of our experience of the intersection of convex solids—without the introduction of any new experience or concept not derivable from that experience of intersection? If such a definition is possible, how are we to proceed to discover it? In the third place, how are we to tell when three or more of our new points are situated on a single line, and how are we to define such a line? These three problems will form the chief subject-matter of the remainder of this lecture and of the following lecture.

Let us take them up in the order just indicated. How, we asked, is it possible to get along in a system in which some points—those within the region directly accessible to our experience—are the elements of which lines are made up as classes, while the remainder of the points of space are classes of lines? The answer is—it is not possible, and we do not intend to try to do so in this paper. Not only is it highly inconvenient and unnatural for one

point in a system of geometry to be an aggregate of aggregates of other points, but there are good philosophical reasons—indicated by Mr. Russell in that part of the *Principia Mathematica* which deals with the Theory of Types, but too complicated and foreign to the subject-matter of this course of lectures for us to discuss here—there are good philosophical reasons, I say, for holding that no assertion which can be made significantly concerning a given entity, say x, can also be made significantly concerning a collection of collections which has some member of which x is in turn a member. Therefore, since a line, in the sense defined in our last lecture, is a class of the points which we then defined, it is impossible for one to assert any proposition significantly concerning these points, on the one hand, and also concerning the classes of lines that we intend in the future to call points, on the other. Now it is, to say the least, extremely awkward to have to phrase every proposition that concerns itself with points in one manner when it concerns itself with the points inside a given region and in an entirely different manner when it deals with the points outside this region. We shall consequently define the points inside the region directly accessible to experience as well as those outside it as classes of the lines that we defined in our last lecture, and we shall term all points *qua* classes of straight lines *generalized points,* in order that we may not confuse them with the points defined in our last lecture. Just as we agreed to regard each of the generalized points lying beyond the bounds of our direct experience as the class of all the lines which, as we should say in every-day language, are directed towards the point, so we shall agree to regard those generalized points lying within the region directly accessible to our experience as a class of all the lines which we should usually consider to pass through some point inside this region. If such a class of lines happens to be the class of all the lines which con-

tain in common a point, in the sense defined in our last lecture, then if x be the point which they all have in common and a be the class of lines, we shall say that a is the generalized point corresponding to x, but it must be clearly understood that *a is not x.*

We are now in a position to deal with our second question: is it possible to define the property which a class of lines has when every member of it, as we should usually put it, is directed towards a given point, in terms of our experience of the intersection of convex solids? It should be noticed that this is the crucial question of this entire lecture, and that our reduction of generalized points to classes of lines having some property which we should usually call "passing through a given point," but which, as a matter of fact, we wish to define without reference to any point through which the lines are supposed to pass, is in the unpleasant situation of Mahomet's coffin until we find a way of identifying this property. In the analogous instance of the astronomers and the star or planet, a collection of telescopes all pointing at a certain point in space is, as we saw, distinguished from a collection of telescopes not all pointing at any one point in space by the fact that when all the telescopes are directed towards a single point certain trigonometrical formulae connecting the latitudes, longitudes, geographical directions, and slopes of the several telescopes hold good which do not hold good when the telescopes are not all aimed at any one point. Such a method of determining whether or not all the lines of a given collection are directed to a single point is inapplicable to the case where we are to define the generalized points of space solely in terms of its points and lines in the sense of our last lecture, for we have as yet no definition of an angle or of a distance nor of a slope: all that we have defined up to this point is the set of all the points and linear segments that lie within a given region. Our problem

hence reduces itself to that of the determination of such classes of lines as are made up from all the lines that pass through a given region—the region, namely, within which we experience the intersection of convex solids—and some chosen point inside or outside of this region, in terms of the intersection-relations of those portions of lines lying inside the region.

We can, however, narrow our problem still further, and indicate the method of its solution with still greater definiteness, if we remember a certain fact about straight lines which we pointed out when we were discussing the case where several observers are looking at a single star. It will be remembered that we showed that if the axes of the telescopes *a, b*, and *c* converge to a single point and the axes of the telescopes *b, c,* and *d* likewise converge to a single point then the axes of all the four telescopes *a, b, c,* and *d* all converge to the *same* point. As we may easily show by a diagram, we may generalize this statement and say that, given any collection of telescopes, if there are two among them, say *a* and *b,* such that if *x* be any member of the collection of telescopes the axes of *a*, *b,* and *x* all converge to one point, then all the axes of the telescopes of the collection converge to a single point. The converse of this statement is even more obviously true. We can thus define a collection of telescopes as one, the axes of all of whose members converge to some one point provided that it contains two members the axes of which intersect and that the collection is made up of all telescopes which form, taken together with these two, a triad of telescopes whose axes converge to a single point, and in addition of these two telescopes themselves. If, that is, we are in possession of a criterion of the convergence of a triad of telescope-axes, we are able to define the collections of all telescope-axes converging to some point or other. In exactly the same manner we are able to define certain classes of the

lines we discussed in our last lecture as classes of all the lines which, we should ordinarily say, pass through some point or other, whether that point is or is not within that region which is directly accessible to our experience, or in other words, as generalized points. If we have a criterion which enables us to discover when any three lines converge to any point whatsoever in space, for the property of line-triads which reads, "If two line-triads each of which is made up of three lines converging to a point possess two lines in common, all four lines making up the two triads pass through some single point," is not confined in its application to the axes of telescopes but applies equally well to all kinds of lines. Therefore, if we are already in the possession of a definition of a convergent triad of lines, we may define a generalized point as the class of all lines, in the sense in which we defined lines in our last lecture, which either are one of two given lines, say l and m, or form together with l and m triads in which the three members of the triad stand to one another in the relation which is ordinarily denominated 'all passing through the same point,' provided only that l and m are distinct intersecting lines—that is, distinct lines which form two of the members of some triad of lines which all would naturally be said to pass through some point. If, then, we are able to give a definition of the relation among three portions of lines lying inside a given region of space which we should naturally call, that of all being directed towards some one point and which the mathematician terms the relation of concurrence, which involves only such notions as we can define in terms of the points and lines of our last lecture, we are in a position to define the class of all generalized points in space, wherever they may be situated. One of the notions which it is permitted for us to use in the definition of the concurrence of three lines is that of the relation which two of the lines of our last lecture bear to one another

when they possess in common one of the points of our last lecture as a member, for this notion can be defined in terms of points, lines, and notions of pure logic alone, and consequently ultimately in terms of the experience of the intersection of convex solids and of in addition only such notions as belong to pure logic and not to concrete experience.

This demand will be satisfied if we give an adequate definition of the concurrence of any three lines which do not all lie in a single plane, for we can define in terms of this relation the concurrence of any three lines whatever, whether they are concurrent or not, in the following manner: the lines *a, b,* and *c* are said to be concurrent if *d* and *e* are two lines such that each of the three lines *a, b,* and *c* forms with *d* and *e* a triad whose members are concurrent *and do not lie in the same plane.* Now, it is an extremely easy task to give a definition of a relation between three lines not all in the same plane, which, though it is slightly more general than the relation of concurrence, includes the latter as a special case, and is only *slightly* more general than it. This new relation is that which holds among three coplanar lines when they are either all concurrent or all parallel. We shall introduce into the definition of this no notion which we have not already defined in terms of the experience that we have taken as fundamental—the experience, namely, of the intersection of convex solids. We can define a plane, readily enough, as the class of all those points, in the sense in which we have already defined points, which lie on any line which has two distinct points in common with some given pair of lines that themselves have a point in common, but do not coincide. As the lines that we have already defined really represent those segments of the lines of ordinary geometry intercepted by the surface of that region of space within which convex solids appear to intersect, our planes, as they are now defined, will actu-

ally represent the planar areas intercepted by the surface of this region, provided that it is possible to draw from any point of such an area a line cutting any two given linear segments intercepted by the surface of the area in two distinct points. That this is possible we may readily show to be the case under the hypothesis, which we have every reason to believe satisfied, that the region of space accessible to our experience is convex. Now, it is a familiar theorem of elementary solid geometry that if p, q, and r be any three distinct planes of which no two are parallel and which do not all possess any line in common, then the intersection of p and q, the intersection of q and r, and the intersection of r and p will form a triad of concurrent or parallel lines. The proof of this theorem is simple, and the situation it represents is illustrated by the corner of a room, where the walls and floor represent p, q, and r, and the three edges of the room that meet at the corner are the lines of intersection of pairs of the planes p, q, and r. The case where the three lines are parallel is represented by the three faces and the three edges of a triangular prism. From these examples, it is further easy to guess the truth of the converse theorem of that which we have just stated: three lines not all in one plane are concurrent or parallel *only* when they are the three lines of intersection of pairs of the planes belonging to a certain triad. If we apply these theorems to the lines of our last lecture, under the hypothesis that these represent the linear segments intercepted by the surface of a certain convex region, we shall obtain the result that three of the lines of our last lecture that do not all lie in one plane are concurrent, wherever the point of their concurrence may be situated, or parallel *when and only when they are coplanar by pairs.* Since the relation of coplanarity among the lines of our last lecture has been already defined and the concurrence or parallelism of three lines not all coplanar has not yet been defined in terms of

our experience of the intersection of convex solids, we may regard the equivalence expressed in our last sentence as a *definition* of the concurrence or parallelism of three lines not all coplanar.

We may now go on and say that three of our lines, l, m, and n are concurrent or parallel, whether they are all coplanar or not, when and only when they are all distinct and there are two of our lines, a and b, let us say, which are such that a, b, and l, a, b, and m, and a, b, and n form three triads, respectively, each made up of three concurrent or parallel lines that are not all coplanar, in the manner that we just defined in the last paragraph. This definition *resembles* the definition of the concurrence of three lines, whether they are all in the same plane or not, which we suggested earlier in this lecture, but it differs from the latter in that it defines the *concurrence or parallelism* of any three lines, and not their simple concurrence. To prove the adequacy of this definition is a simple matter, and reduces itself to the proof that if a and b intersect, and l, m, and n each form with a and b a triad of concurrent lines, l, m, and n are concurrent, and that further if a and b are parallel, and l, m, and n each form with a and b a triad of parallel lines, then l, m, and n are all parallel to one another. These two propositions are obvious on inspection. We thus see that our definition of parallel or concurrent triads of lines covers those triads, and only those triads, of the lines of our last lecture that we should naturally call concurrent or parallel triads.

To sum up what we have said in this lecture: we saw that the definitions of our last lecture yield us only those points and linear segments within a certain limited region of space. We found it necessary, therefore, to search for a definition of all the points and lines of space in terms of those lying inside this region, and found our problem analogous to that of the astronomer in the location of a

planet by its parallax. We learned from this example that if we were in possession of a definition of the concurrence of three lines, we could define a point anew as a class of concurrent lines of the sort defined in our last lecture, and thus obtain a system of points extending throughout space. We searched for such a definition of the concurrence of three lines, but found instead a definition of the concurrence or parallelism of three of the lines of our last lecture, involving no concrete notion other than that of our experience of the intersection of convex solids. The problems that remain before us in the next lecture are first, that of observing what effects the difference between the relation of concurrence, for which we sought, and that of concurrence or parallelism, which we obtained, will involve with respect to our new points and their definition, and secondly, that of the definition of the lines that connect our new points.

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